# Affine Hecke algebras and the Schubert calculus

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*Dedicated to Alain Lascoux*

## 0. Introduction

Using a combinatorial approach which avoids geometry, this paper studies the ring structure of $K_T(G/B)$, the $T$-equivariant $K$-theory of the (generalized) flag variety $G/B$. Here, the data $G \supseteq B \supseteq T$ is a complex reductive algebraic group (or symmetrizable Kac-Moody group) $G$, a Borel subgroup $B$, and a maximal torus $T$, and $K_T(G/B)$ is the Grothendieck group of $T$-equivariant coherent sheaves on $G/B$. Because of the $T$-equivariance the ring $K_T(G/B)$ is an $R$-algebra, where $R$ is the representation ring of $T$. As explained by Grothendieck [Gd] (in the non Kac-Moody case) and Kostant and Kumar [KK] (in the general Kac-Moody case), the ring $K_T(G/B)$ has a natural $R$-basis $\{[\mathcal{O}_{X_w}] \mid w \in W\}$, where $W$ is the Weyl group and $\mathcal{O}_{X_w}$ is the structure sheaf of the Schubert variety $X_w \subseteq G/B$. One of the main problems in the field is to understand the structure constants of the ring $K_T(G/B)$ with this basis, that is, the coeffients $c^z_{wv}$ in the equations

$$[\mathcal{O}_{X_w}][\mathcal{O}_{X_v}] = \sum_{z \in W} c^z_{wv} [\mathcal{O}_{X_z}]. \tag{0.1}$$

Our approach is to work completely combinatorially and define $K_T(G/B)$ as a quotient of the affine nil-Hecke algebra. The fact that the combinatorial approach coincides with the geometric one is a consequence of the results of Kostant and Kumar [KK] and Demazure [D]. In the combinatorial literature the elements $[\mathcal{O}_{X_w}]$ are often called (double) Grothendieck polynomials.

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Let $P$ be the weight lattice of $G$ and, for $\lambda \in P$, let $[X^\lambda]$ be the homogeneous line bundle on $G/B$ corresponding to the character of $T$ indexed by $\lambda$. The theorem of Pittie [P] says that the ring $K_T(G/B)$ is generated by the $[X^\lambda]$, $\lambda \in P$. Steinberg [St] strengthened this result by displaying specific $[X^{-\lambda_w}]$, $w \in W$, which form an $R$-basis of $K_T(G/B)$. These results are often collectively known as the "Pittie-Steinberg theorem".

The theorems which we prove in Section 2 are simply different points of view on the Pittie-Steinberg theorem. Though we are not aware of any reference which states these theorems in the generality which we consider, these theorems should be considered well known.

Let $s_1, \ldots, s_n$ be the simple reflections in $W$ (determined by the data $(G \supseteq B \supseteq T)$), let $w_0$ be the longest element of $W$ and let $P^+$ be the set of dominant weights in $P$. The Schubert varieties $X_{w_0 s_i}$ are the codimension one Schubert varieties in $G/B$. In section 3 we prove "Pieri-Chevalley" formulas for the products

$$[X^\lambda][\mathcal{O}_{X_w}], \qquad [X^{-\lambda}][\mathcal{O}_{X_w}], \qquad [X^{w_0\lambda}][\mathcal{O}_{X_w}], \qquad \text{and} \qquad [\mathcal{O}_{X_{w_0 s_i}}][\mathcal{O}_{X_w}], \tag{0.2}$$

for $\lambda \in P^+$, $w \in W$ and $1 \leq i \leq n$. All of these Pieri-Chevalley formulas are given in terms of the combinatorics of the Littelmann path model [Li1-3]. The formula which we give for the first product in (0.2) is due to Pittie and Ram [PR1]. In this paper we provide more details of proof than appeared in [PR1]. The other formulas for the products in (0.2) follow by applying the duality theorem of Brion [Br, Theorem 4] to the first formula. However, here we give an independent, combinatorial, proof and deduce Brion's result as a consequence. The last formula is a consequence of the nice formula

$$[\mathcal{O}_{X_{w_0 s_i}}] = 1 - e^{w_0\omega_i}[X^{-\omega_i}], \tag{0.3}$$

which is an easy consequence of the first two Pieri-Chevalley rules.

It is not difficult to "specialize" product formulas for $K_T(G/B)$ to corresponding product formulas for $K(G/B)$, $H^*_T(G/B)$, and $H^*(G/B)$ (by using the Chern character and comparing lowest degree terms, and ignoring the $T$-action). Thus the products which are computed in this paper also give results for ordinary Grothendieck polynomials, double Schubert polynomials, and ordinary Schubert polynomials. In section 4 we explain how to do these conversions. For most of these cases the specialized versions of our Pieri-Chevalley rules are already very well known (see, for example, [Ch]).

In Section 5 we give explicitly

(a) two different kinds of formulas for $[\mathcal{O}_{X_w}]$ in terms of $X^\lambda$, and

(b) complete computations of the products in (0.1)

for the rank two root systems. This data allows us to make a "positivity conjecture" for the coefficients $c^z_{wv}$ in (0.1). This conjecture generalizes the theorems of Brion [Br, formula before Theorem 1] and Graham [Gr, Corollary 4.1], which treat the cases $K(G/B)$ and $H^*_T(G/B)$, respectively.

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## 1. Preliminaries

Fix the following data and notation:

| | |
|---|---|
| $\mathfrak{h}^*$ | is a real vector space of dimension $n$, |
| $R$ | is a reduced irreducible root system in $\mathfrak{h}^*$, |
| $R^+$ | is a set of positive roots in $R$, |
| $W$ | is the Weyl group of $R$, |
| $s_1, \ldots, s_n$ | are the simple reflections in $W$, |
| $m_{ij}$ | is the order of $s_i s_j$ in $W$, $i \neq j$, |
| $R(w) = \{\alpha \in R^+ \mid w\alpha \notin R^+\}$ | is the inversion set of $w \in W$, |
| $\ell(w) = \mathrm{Card}(R(w))$ | is the length of $w \in W$, |
| $\leq$ | is the Bruhat-Chevalley order on $W$, |
| $\alpha_1, \ldots, \alpha_n$ | are the simple roots in $R^+$, |
| $\omega_1, \ldots, \omega_n$ | are the fundamental weights, |
| $P = \sum_{i=1}^n \mathbb{Z}\omega_i$ | is the weight lattice, |
| $P^+ = \sum_{i=1}^n \mathbb{Z}_{\geq 0}\omega_i$ | is the set of dominant integral weights. |

For a brief, easy, introduction to root systems with lots of pictures for visualization see [NR]. By [Bou VI §1 no. 6 Cor. 2 to Prop. 17], if $w = s_{i_1} \cdots s_{i_p}$ be a reduced word for $w$, then

$$R(w) = \{\alpha_{i_p}, s_{i_p}\alpha_{i_{p-1}}, \ldots, s_{i_p} \cdots s_{i_2}\alpha_{i_1}\}, \tag{1.1}$$

The *affine nil-Hecke algebra* is the algebra $\tilde{H}$ given by generators $T_1, \ldots, T_n$ and $X^\lambda$, $\lambda \in P$, with relations

$$T_i^2 = T_i, \qquad \underbrace{T_iT_jT_i\cdots}_{m_{ij}\ \text{factors}} = \underbrace{T_jT_iT_j\cdots}_{m_{ij}\ \text{factors}}, \qquad X^\lambda X^\mu = X^{\lambda+\mu}, \tag{1.2}$$

and

$$X^\lambda T_i = T_i X^{s_i\lambda} + \frac{X^\lambda - X^{s_i\lambda}}{1 - X^{-\alpha_i}}. \tag{1.3}$$

Let $T_w = T_{i_1} \cdots T_{i_p}$ for a reduced word $w = s_{i_1} \cdots s_{i_p}$. Then

$$\{X^\lambda T_w \mid w \in W, \lambda \in P\} \qquad \text{and} \qquad \{T_w X^\lambda \mid w \in W, \lambda \in P\} \tag{1.4}$$

are bases of $\tilde{H}$.

Both the *nil-Hecke algebra*,

$$H = \mathbb{Z}\text{-span}\{T_w \mid w \in W\}, \qquad \text{and} \qquad \mathbb{Z}[X] = \mathbb{Z}\text{-span}\{X^\lambda \mid \lambda \in P\} \tag{1.5}$$

are subalgebras of $\tilde{H}$. The action of $W$ on $\mathbb{Z}[X]$ is given by defining

$$wX^\lambda = X^{w\lambda}, \qquad \text{for } w \in W,\ \lambda \in P, \tag{1.6}$$

and extending linearly. The proof of the following theorem is given in [R, Theorem 1.13 and Theorem 1.17]. The first statement of the theorem is due to Bernstein, Zelevinsky, and Lusztig [Lu, 8.1] and the second statement is due to Steinberg [St] and is known as the Pittie-Steinberg theorem.

**Theorem 1.7.** *Define*

$$\lambda_w = w^{-1} \sum_{s_i w < w} \omega_i, \qquad \text{for } w \in W. \tag{1.8}$$

*The center of $\tilde{H}$ is $Z(\tilde{H}) = \mathbb{Z}[X]^W$ and each element $f \in \mathbb{Z}[X]$ has a unique expansion*

$$f = \sum_{w \in W} f_w X^{-\lambda_w}, \qquad \text{with } f_w \in \mathbb{Z}[X]^W. \tag{1.9}$$

Let $\varepsilon_i = 1 - T_i$ and let $\varepsilon_w = \varepsilon_{i_1} \cdots \varepsilon_{i_p}$ for a reduced word $w = s_{i_1} \cdots s_{i_p}$. Then $\varepsilon_w$ is well defined and independent of the reduced word for $w$ since

$$\varepsilon_i^2 = \varepsilon_i, \qquad \text{and} \qquad \underbrace{\varepsilon_i \varepsilon_j \varepsilon_i \cdots}_{m_{ij} \text{ factors}} = \underbrace{\varepsilon_j \varepsilon_i \varepsilon_j \cdots}_{m_{ij} \text{ factors}}. \tag{1.10}$$

The second equality is a consequence of the formulas

$$\varepsilon_w = \sum_{v \le w} (-1)^{\ell(v)} T_v \qquad \text{and} \qquad T_w = \sum_{v \le w} (-1)^{\ell(v)} \varepsilon_v \tag{1.11}$$

which are straightforward to verify by induction on the length of $w$.

## 2. The ring $K_T(G/B)$

Let $H$ and $\mathbb{Z}[X]$ be as in (1.5). The *trivial representation* of $H$ is defined by the homomorphism $\mathbf{1} \colon H \to \mathbb{Z}$ given by $\mathbf{1}(T_i) = 1$. The first of the maps

$$\begin{array}{ccccc} \mathbb{Z}[X] & \xrightarrow{\sim} & \tilde{H} T_{w_0} & \xrightarrow{\sim} & \tilde{H} \otimes_H \mathbf{1} \\ f & \longmapsto & f T_{w_0} & \longmapsto & f \otimes \mathbf{1} \end{array}$$

is an $\tilde{H}$-module isomorphism if the action of $\tilde{H}$ on $\mathbb{Z}[X]$ is given by

$$T_i \cdot f = \frac{X^{\alpha_i} f - s_i f}{X^{\alpha_i} - 1}, \qquad \text{for } f \in \mathbb{Z}[X]. \tag{2.1}$$

The group algebra of $P$ is

$$R = \mathbb{Z}\text{-span}\{e^\lambda \mid \lambda \in P\} \qquad \text{with} \qquad e^\lambda e^\mu = e^{\lambda + \mu}, \tag{2.2}$$

for $\lambda, \mu \in P$. Extend coefficients to $R$ so that $\tilde{H}_R = R \otimes_{\mathbb{Z}} \tilde{H}$ and $R[X] = R \otimes_{\mathbb{Z}} \mathbb{Z}[X]$ are $R$-algebras. Define $K_T(G/B)$ to be the $\tilde{H}_R$-module

$$K_T(G/B) = R\text{-span}\{[\mathcal{O}_{X_w}] \mid w \in W\}, \tag{2.3}$$

so that the $[\mathcal{O}_{X_w}]$, $w \in W$, are an $R$-basis of $K_T(G/B)$, with $\tilde{H}_R$-action given by

$$X^\lambda [\mathcal{O}_{X_1}] = e^\lambda [\mathcal{O}_{X_1}], \qquad \text{and} \qquad T_i [\mathcal{O}_{X_w}] = \begin{cases} [\mathcal{O}_{X_{ws_i}}], & \text{if } ws_i > w, \\ [\mathcal{O}_{X_w}], & \text{if } ws_i < w. \end{cases} \tag{2.4}$$

If $R$ is an $R[X]$-module via the $R$-algebra homomorphism given by

$$e\colon \begin{array}{ccc} R[X] & \longrightarrow & R \\ X^\lambda & \longmapsto & e^\lambda \end{array} \tag{2.5}$$

then, as $\tilde{H}_R$-modules, $K_T(G/B) \cong \tilde{H}_R \otimes_{R[X]} R_e$, where $R_e$ is the $R$-rank 1 $R[X]$-module determined by the homomorphism $e$.

Let $Q$ be the field of fractions of $R$ and let $\overline{Q}$ be the algebraic closure of $Q$. For $w \in W$ let

$$b_w \quad \text{in } \overline{Q} \otimes_R K_T(G/B) \quad \text{be determined by} \qquad X^\lambda b_w = e^{w\lambda} b_w, \quad \text{for } \lambda \in P. \tag{2.6}$$

If the $b_w$ exist, then they are a $\overline{Q}$-basis of $\overline{Q} \otimes_R K_T(G/B)$ since they are eigenvectors with distinct eigenvalues. If $\tau_i$, $1 \le i \le n$, are the operators on $\bar{Q} \otimes_R K_T(G/B)$ given by

$$\tau_i = T_i - \frac{1}{1 - X^{-\alpha_i}}, \qquad \text{then} \qquad b_1 = [\mathcal{O}_{X_1}] \quad \text{and} \quad \tau_i b_w = b_{ws_i}, \quad \text{for } ws_i > w, \tag{2.7}$$

because, a direct computation with relation (1.3) gives that $X^\lambda \tau_i b_w = \tau_i X^{s_i\lambda} b_w = \tau_i e^{ws_i\lambda} b_w = e^{ws_i\lambda} b_{ws_i}$. Thus the $b_w$, $w \in W$, exist and the form of the $\tau$-operators shows that, in fact, they form a $Q$-basis of $Q \otimes_R K_T(G/B)$ (it was not really necessary to extend coefficients all the way to $\overline{Q}$). Equations (2.6) and (2.7) force

$$\underbrace{\tau_i\tau_j\tau_i\cdots}_{m_{ij}\text{ factors}} = \underbrace{\tau_j\tau_i\tau_j\cdots}_{m_{ij}\text{ factors}}, \qquad \text{and the equality} \qquad \tau_i^2 = \frac{1}{(X^{\alpha_i}-1)(X^{-\alpha_i}-1)}$$

is checked by direct computation using (1.3). Let $\tau_w = \tau_{i_1}\cdots\tau_{i_p}$ for a reduced word $w = s_{i_1}\cdots s_{i_p}$. Then, for $w \in W$,

$$b_w = \tau_{w^{-1}} b_1, \qquad [\mathcal{O}_{X_w}] = T_{w^{-1}}[\mathcal{O}_{X_1}] \qquad \text{and we define} \qquad [\mathcal{I}_{X_w}] = \varepsilon_{w^{-1}}[\mathcal{O}_{X_1}], \tag{2.8}$$

where $\varepsilon_w$ is as in (1.11). In terms of geometry, $[\mathcal{O}_{X_w}]$ is the class of the structure sheaf of the Schubert variety $X_w$ in $G/B$ and, up to a sign, $[\mathcal{I}_{X_w}]$ is class of the sheaf $\mathcal{I}_{X_w}$ determined by the exact sequence $0 \to \mathcal{I}_{X_w} \to \mathcal{O}_{X_w} \to \mathcal{O}_{\partial X_w} \to 0$, where $\partial X_w = \bigsqcup_{v<w} BvB$ (see [Ma, Theorem 2.1(ii)] and [LS, equation (4)]. We are not aware of a good geometric characterization of the basis $\{[X^{-\lambda_w}] \mid w \in W\}$ of $K_T(G/B)$ which appears in the following theorem.

**Theorem 2.9.** *Let $\lambda_w$, $w \in W$, be as defined in Theorem 2.9 and let $[X^\lambda] = X^\lambda[\mathcal{O}_{X_{w_0}}] = X^\lambda T_{w_0}[\mathcal{O}_{X_1}]$ for $\lambda \in P$. Then the $[X^{-\lambda_w}]$, $w \in W$, form an $R$-basis of $K_T(G/B)$.*

*Proof.* Up to constant multiples, $[\mathcal{O}_{X_{w_0}}] = T_{w_0}[\mathcal{O}_{X_1}]$ is determined by the property

$$T_i[\mathcal{O}_{X_{w_0}}] = [\mathcal{O}_{X_{w_0}}], \qquad \text{for all } 1 \le i \le n. \tag{2.10}$$

If constants $c_w \in Q$ are given by

$$[\mathcal{O}_{X_{w_0}}] = \sum_{w\in W} c_w b_w,$$

then comparing coefficients of $b_{ws_i}$, for $ws_i > w$, on each side of (2.10) yields a recurrence relation for the $c_w$,

$$c_w = c_{ws_i}\left(\frac{1}{1-e^{-w\alpha_i}}\right) \quad \text{for } ws_i > w, \qquad \text{which implies} \qquad c_{w_0v^{-1}} = \prod_{\alpha\in R(v)} \frac{1}{1-e^{w_0\alpha}}, \tag{2.11}$$

via (1.1) and the fact that $c_{w_0} = 1$. Thus,

$$[X^{-\lambda_v}] = X^{-\lambda_v}[\mathcal{O}_{X_{w_0}}] = \sum_{w\in W} c_w e^{-w\lambda_v} b_w,$$

and if $C$, $M$ and $A$ are the $|W|\times|W|$ matrices given by

$$C = \mathrm{diag}(c_w), \quad M = (e^{-w\lambda_v}), \quad \text{and} \quad A = (a_{zw}), \qquad \text{where} \quad b_w = \sum_{z\in W} a_{zw}[\mathcal{O}_{X_z}],$$

then the transition matrix between the $X^{-\lambda_v}$ and the $[\mathcal{O}_{X_z}]$ is the product $ACM$. By (2.8) and the definition of the $\tau_i$, the matrix $A$ has determinant 1. Using the method of Steinberg [St] and subtracting row $e^{-s_\alpha w\lambda_v}$ from row $e^{-w\lambda_v}$ in the matrix $M$ allows one to conclude that $\det(M)$ is divisible by

$$\prod_{\alpha\in R^+}(1-e^{-\alpha})^{|W|/2} \qquad \text{and identifying} \quad \prod_{w\in W} e^{-w\lambda_w} = \prod_{i=1}^{n}\prod_{s_iw<w} e^{-\omega_i} = (e^{-\rho})^{|W|/2}$$

as the lowest degree term determines $\det(M)$ exactly. Thus,

$$\det(ACM) = 1\cdot\left(\prod_{w\in W}\prod_{\alpha\in R(w)}\frac{1}{1-e^{-\alpha}}\right)\left(e^{\rho}\prod_{\alpha\in R^+}(1-e^{-\alpha})\right)^{|W|/2} = (e^{\rho})^{|W|/2}.$$

Since this is a unit in $R$, the transition matrix between the $[\mathcal{O}_{X_w}]$ and the $X^{-\lambda_v}$ is invertible. ∎

**Theorem 2.12.** *The composite map*

$$\begin{array}{rccccccc} \Phi\colon & R[X] & \longrightarrow & \tilde{H}_R T_{w_0} & \hookrightarrow & \tilde{H}_R & \longrightarrow & K_T(G/B) \\ & f & \longmapsto & fT_{w_0} & & h & \longmapsto & h[\mathcal{O}_{X_1}] \end{array}$$

*is surjective with kernel*

$$\ker\Phi = \langle f - e(f) \mid f\in R[X]^W\rangle,$$

*the ideal of the ring $R[X]$ generated by the elements $f-e(f)$ for $f\in R[X]^W$. Hence*

$$K_T(G/B) \;\cong\; \frac{R[X]}{\langle f-e(f)\mid f\in R[X]^W\rangle}$$

*has the structure of a ring.*

*Proof.* Since $\Phi(X^\lambda) = X^\lambda T_{w_0}[\mathcal{O}_{X_1}] = X^\lambda[\mathcal{O}_{X_{w_0}}]$, it follows from Theorem 2.9 that $\Phi$ surjective. Thus $K_T(G/B)\cong R[X]/\ker\Phi$. Let $I = \langle f-e(f)\mid f\in R[X]^W\rangle$. If $f\in R[X]^W$ then, for all $\lambda\in P$,

$$\begin{aligned}\Phi(X^\lambda(f-e(f))) &= X^\lambda(f-e(f))T_{w_0}[\mathcal{O}_{X_1}] = X^\lambda T_{w_0}(f-e(f))[\mathcal{O}_{X_1}] \\ &= X^\lambda T_{w_0}(e(f)-e(f))[\mathcal{O}_{X_1}] = 0,\end{aligned}$$

since $f-e(f)\in Z(\tilde{H}_R)$. Thus $I\subseteq\ker\Phi$. The ring $K_T(G/B) = R[X]/\ker\Phi$ is a free $R$-module of rank $|W|$ and, by Theorem 1.7, so is $R[X]/I$. Thus $\ker\Phi = I$. ∎

## 3. Pieri-Chevalley formulas

Recall that both

$$\{X^\lambda T_{w^{-1}} \mid \lambda \in P, w \in W\} \qquad \text{and} \qquad \{T_{z^{-1}} X^\mu \mid \mu \in P, z \in W\} \qquad \text{are bases of } \tilde{H}.$$

If $c^{\mu,z}_{w,\lambda} \in \mathbb{Z}$ are the entries of the transition matrix between these two bases,

$$X^\lambda T_{w^{-1}} = \sum_{z \in W, \mu \in P} c^{\mu,z}_{w,\lambda} T_{z^{-1}} X^\mu, \tag{3.1}$$

then applying each side of (3.1) to $[\mathcal{O}_{X_1}]$ gives that

$$[X^\lambda][\mathcal{O}_{X_w}] = \sum_{z \in W, \mu \in P} c^{\mu,z}_{w,\lambda} e^\mu [\mathcal{O}_{X_z}], \qquad \text{in } K_T(G/B).$$

This is the most general form of "Pieri-Chevalley rule". The problem is to determine the coefficients $c^{\mu,z}_{w,\lambda}$.

*The path model*

A *path* in $\mathfrak{h}^*$ is a piecewise linear map $p\colon [0,1] \to \mathfrak{h}^*$ such that $p(0) = 0$. For each $1 \le i \le n$ there are *root operators* $e_i$ and $f_i$ (see [L3] Definitions 2.1 and 2.2) which act on the paths. If $\lambda \in P^+$ the *path model* for $\lambda$ is

$$\mathcal{T}^\lambda = \{f_{i_1} f_{i_2} \cdots f_{i_l} p_\lambda\},$$

the set of all paths obtained by applying the root operators to $p_\lambda$, where $p_\lambda$ is the straight path from 0 to $\lambda$, that is, $p_\lambda(t) = t\lambda$, $0 \le t \le 1$. Each path $p$ in $\mathcal{T}^\lambda$ is a concatenation of segments

$$p = p^{a_1}_{w_1\lambda} \otimes p^{a_2}_{w_2\lambda} \otimes \cdots \otimes p^{a_r}_{w_r\lambda} \qquad \text{with} \qquad w_1 \ge w_2 \ge \cdots \ge w_r \quad \text{and} \quad a_1 + a_2 + \cdots + a_r = 1, \tag{3.2}$$

where, for $v \in W$ and $a \in (0,1]$, $p^a_{v\lambda}$ is a piece of length $a$ from the straight line path $p_{v\lambda} = vp_\lambda$. If $W_\lambda = \mathrm{Stab}(\lambda)$ then the $w_j$ should be viewed as cosets in $W/W_\lambda$ and $\ge$ denotes the order on $W/W_\lambda$ inherited from the Bruhat-Chevalley order on $W$. The total length of $p$ is the same as the total length of $p_\lambda$ which is assumed (or normalized) to be 1. For $p \in \mathcal{T}^\lambda$ let

$$\begin{aligned} p(1) &= \sum_{i=1}^r a_i w_i \lambda \quad \text{be the endpoint of } p, \\ \iota(p) &= w_1, \quad \text{the initial direction of } p, \quad \text{and} \\ \phi(p) &= w_r, \quad \text{the final direction of } p. \end{aligned}$$

If $h \in \mathcal{T}^\lambda$ is such that $e_i(h) = 0$ then $h$ is the *head* of its *i-string*

$$S^\lambda_i(h) = \{h, f_i h, \ldots, f_i^m h\},$$

where $m$ is the smallest positive integer such that $f_i^m h \ne 0$ and $f_i^{m+1} h = 0$. The full path model $\mathcal{T}^\lambda$ is the union of its $i$-strings. The endpoints and the inital and final directions of the paths in the $i$-string $S^\lambda_i(h)$ have the following properties:

$$\begin{array}{ll} (f_i^k h)(1) & = h(1) - k\alpha_i, \qquad \text{for } 0 \le k \le m, \\ & \\ \text{either} & \iota(h) = \iota(f_i h) = \cdots = \iota(f_i^m h) < s_i \iota(h) \\ \text{or} & \iota(h) < \iota(f_i h) = \cdots = \iota(f_i^m h) = s_i \iota(h), \qquad \text{and} \\ & \\ \text{either} & s_i \phi(f_i^m h) < \phi(h) = \cdots = \phi(f_i^{m-1} h) = \phi(f_i^m h) \\ \text{or} & s_i \phi(f_i^m h) = \phi(h) = \cdots = \phi(f_i^{m-1} h) < \phi(f_i^m h). \end{array} \tag{3.3}$$

The first property is [L2] Lemma 2.1a, the second is is [L1] Lemma 5.3, and the last is a result of applying [L2] Lemma 2.1e to [L1] Lemma 5.3. All of these facts are really coming from the explicit form of the action of the root operators on the paths in $\mathcal{T}^\lambda$ which is given in [L1] Proposition 4.2.

Let $\lambda \in P^+$, $w \in W$ and $z \in W/W_\lambda$, and let $p \in \mathcal{T}^\lambda$ be such that $\iota(p) \le wW_\lambda$ and $\phi(p) \ge z$. Write $p$ in the form (3.2) and let $\tilde w_1, \ldots, \tilde w_r, \tilde z$ be the maximal (in Bruhat order) coset representatives of the cosets $w_1, \ldots, w_r, z$ such that

$$w \ge \tilde w_1 \ge \tilde w_2 \ge \cdots \ge \tilde w_r \ge \tilde z. \tag{3.4}$$

**Theorem 3.5.** *Recall the notation $\varepsilon_v$ from (1.11). Let $\lambda \in P^+$ and let $W_\lambda = \mathrm{Stab}(\lambda)$. Let $w \in W$. Then, in the affine nil-Hecke algebra $\tilde H$,*

$$X^\lambda T_{w^{-1}} = \sum_{\substack{p\in\mathcal{T}^\lambda\\ \iota(p)\le wW_\lambda}} T_{\phi(p)^{-1}} X^{p(1)} \qquad\text{and}\qquad X^\lambda \varepsilon_{w^{-1}} = \sum_{\substack{p\in\mathcal{T}^\lambda\\ \iota(p)=w}} \sum_{\substack{z\in W/W_\lambda\\ z\le\phi(p)}} (-1)^{\ell(w)+\ell(z)} \varepsilon_{\tilde z^{-1}} X^{p(1)},$$

*where, if $W_\lambda \ne \{1\}$ then $T_{\phi(p)^{-1}} = T_{\tilde w_r^{-1}}$ and $\varepsilon_{z^{-1}} = \varepsilon_{\tilde z^{-1}}$ with $\tilde w_r$ and $\tilde z$ as in (3.4).*

*Proof.* (a) The proof is by induction on $\ell(w)$. Let $w = s_i v$ where $s_i v > v$. Define

$$\mathcal{T}^\lambda_{\le w} = \{p \in \mathcal{T}^\lambda \mid \iota(p) \le wW_\lambda\}.$$

Assume $w = s_i v > v$. Then the facts in (3.3) imply that
(1) $\mathcal{T}^\lambda_{\le w}$ is a union of the strings $S_i(h)$ such that $h \in \mathcal{T}^\lambda_{\le v}$, and
(2) If $h \in \mathcal{T}^\lambda_{\le v}$ then either $S_i(h) \subseteq \mathcal{T}^\lambda_{\le v}$ or $S_i(h) \cap \mathcal{T}^\lambda_{\le v} = \{h\}$.

Using the facts in (3.3), a direct computation with the relation (1.3) establishes that, if $h \in \mathcal{T}^\lambda_{\le v}$ then

$$\sum_{p\in S_i(h)} T_{\phi(p)^{-1}} X^{\eta(1)} = T_{\phi(h)^{-1}} X^{h(1)} T_i, \qquad \text{and}$$

$$\sum_{p\in S_i(h)} T_{\phi(p)^{-1}} X^{\eta(1)} = \begin{cases} T_{\phi(h)^{-1}} X^{h(1)} T_i, & \text{if } S_i(h) \subseteq \mathcal{T}^\lambda_{\le v}, \\ T_{\phi(h)^{-1}} X^{h(1)} T_i, & \text{if } S_i(h) \cap \mathcal{T}^\lambda_{\le v} = \{h\}. \end{cases}$$

Thus

$$\begin{aligned}
X^\lambda T_{w^{-1}} = X^\lambda T_{v^{-1}} T_i &= \left( \sum_{p\in\mathcal{T}^\lambda_{\le v}} T_{\phi(p)^{-1}} X^{p(1)} \right) T_i \qquad\qquad \text{(by induction)} \\
&= \sum_{\substack{h\in\mathcal{T}^\lambda_{\le v}\\ e_i(h)=0}} \left( \sum_{S_i(h)\subseteq\mathcal{T}^\lambda_{\le v}} \sum_{p\in S_i(h)} T_{\phi(p)^{-1}} X^{p(1)} + \sum_{S_i(h)\cap\mathcal{T}^\lambda_{\le v}=\{h\}} T_{\phi(h)^{-1}} X^{h(1)} \right) T_i \\
&= \sum_{\substack{h\in\mathcal{T}^\lambda_{\le w}\\ e_i(h)=0}} \left( \sum_{S_i(h)\subseteq\mathcal{T}^\lambda_{\le v}} T_{\phi(h)^{-1}} X^{h(1)} T_i + \sum_{S_i(h)\cap\mathcal{T}^\lambda_{\le v}=\{h\}} T_{\phi(h)^{-1}} X^{h(1)} \right) T_i \\
&= \sum_{\substack{h\in\mathcal{T}^\lambda_{\le w}\\ e_i(h)=0}} \left( \sum_{S_i(h)\subseteq\mathcal{T}^\lambda_{\le v}} T_{\phi(h)^{-1}} X^{h(1)} T_i + \sum_{S_i(h)\cap\mathcal{T}^\lambda_{\le v}=\{h\}} \sum_{p\in S_i(h)} T_{\phi(p)^{-1}} X^{p(1)} \right) \\
&= \sum_{p\in\mathcal{T}^\lambda_{\le w}} T_{\phi(p)^{-1}} X^{p(1)}.
\end{aligned}$$

(b) The proof is similar to case (a). For $w \in W$ let

$$\mathcal{T}^\lambda_{=w} = \{p \in \mathcal{T}^\lambda \mid \iota(p) = wW_\lambda\}.$$

Assume $w = s_i v > v$. Then the facts in (3.3) imply that
(1) $\mathcal{T}^\lambda_{=w}$ is a union of the strings $S_i(h)$ such that $h \in \mathcal{T}^\lambda_{=h}$, and
(2) If $h \in \mathcal{T}^\lambda_{=v}$ then either $S_i(h) \subseteq \mathcal{T}^\lambda_{=v}$ or $S_i(h) \cap \mathcal{T}^\lambda_{=v} = \{h\}$.
Let

$$\mathcal{E}_{\phi(p)} = \sum_{\substack{z \in W/W_\lambda \\ z \le \phi(p)}} (-1)^{\ell(z)} \varepsilon_{\tilde{z}^{-1}}. \tag{3.6}$$

Using (3.3), a direct computation with the relation (1.3) establishes that, if $h \in \mathcal{T}^\lambda_{=v}$ with $e_i h = 0$ then

$$\sum_{p \in S_i(h)} \mathcal{E}_{\phi(p)} X^{p(1)} T_i = 0, \qquad \text{and} \qquad \mathcal{E}_{\phi(h)} X^{h(1)} T_i = -\sum_{p \in S_i(h) - \{h\}} \mathcal{E}_{\phi(p)} X^{p(1)}.$$

Thus

$$\begin{aligned}
X^\lambda \varepsilon_{w^{-1}} = X^\lambda \varepsilon_{v^{-1}} \varepsilon_i &= (-1)^{\ell(v)} \left( \sum_{p \in \mathcal{T}^\lambda_{=v}} \mathcal{E}_{\phi(p)} X^{p(1)} \right) T_i \\
&= (-1)^{\ell(v)} \left( \sum_{S_i(h) \subseteq \mathcal{T}^\lambda_{=v}} \sum_{p \in S_i(h)} \mathcal{E}_{\phi(p)} X^{p(1)} + \sum_{S_i(h) \cap \mathcal{T}^\lambda_{=v} = \{h\}} \mathcal{E}_{\phi(h)} X^{h(1)} \right) T_i \\
&= (-1)^{\ell(v)} \left( 0 - \sum_{S_i(h) \cap \mathcal{T}^\lambda_{=v} = \{h\}} \sum_{p \in S_i(h) - \{h\}} \mathcal{E}_{\phi(p)} X^{p(1)} \right) \\
&= (-1)^{\ell(w)} \left( \sum_{p \in \mathcal{T}^\lambda_{=w}} \mathcal{E}_{\phi(p)} X^{p(1)} \right). \qquad \blacksquare
\end{aligned}$$

**Corollary 3.7.** *Let $\lambda, \mu \in P^+$ and let $w \in W$. Then, in the affine nil-Hecke algebra $\tilde{H}$,*

$$X^{-\lambda} T_{w^{-1}} = \sum_{\substack{p \in \mathcal{T}^{-w_0\lambda} \\ \phi(p) = ww_0}} \sum_{\substack{z \in W/W_{-w_0\lambda} \\ zw_0 \ge \iota(p)}} (-1)^{\ell(w)+\ell(z)} T_{\tilde{z}^{-1}} X^{p(1)} \qquad \textit{and}$$

$$X^{w_0\mu} T_{w^{-1}} = \sum_{\substack{p \in \mathcal{T}^\mu \\ \phi(p) = ww_0}} \sum_{\substack{z \in W/W_\mu \\ zw_0 \le \phi(p)}} (-1)^{\ell(w)+\ell(z)} T_{\tilde{z}^{-1}} X^{p(1)}.$$

*Proof.* The second identity is a restatement of the first with a change of variable $\mu = -w_0\lambda$. The first identity is obtained by applying the algebra involution

$$\begin{array}{ccc} \tilde{H} & \longrightarrow & \tilde{H} \\ T_w & \longmapsto & \varepsilon_w \\ X^\lambda & \longmapsto & X^{-\lambda} \end{array} \qquad \text{and the bijection} \qquad \begin{array}{ccc} \mathcal{T}^\lambda & \longrightarrow & \mathcal{T}^{-w_0\lambda} \\ p & \longrightarrow & p^* \end{array}$$

where $p^*$ is the same path as $p$ except translated so that its endpoint is at the origin. Representation theoretically, this bijection corresponds to the fact that $L(\lambda)^* \cong L(-w_0\lambda)$, if $L(\lambda)$ is the simple $G$-module of highest weight $\lambda$. Note that $p^*(1) = -p(1)$, $\iota(p^*) = \phi(p)w_0$, and $\phi(p^*) = \iota(p)w_0$. ∎

Applying the identities from Theorem 3.5 and Corollary 3.7 to $[\mathcal{O}_{X_1}]$ yields the following product formulas in $K_T(G/B)$. In particular, this gives a combinatorial proof of the ($T$-equivariant extension) of the duality theorem of Brion [Br, Theorem 4]. For $\lambda \in P$ and $w \in W$ let $[X^\lambda] = X^\lambda[\mathcal{O}_{X_{w_0}}] = X^\lambda T_{w_0}[\mathcal{O}_{X_1}]$ and let $c^z_{\lambda,w}$ be given by

$$[X^\lambda][\mathcal{O}_{X_w}] = \sum_{z\in W} c^z_{\lambda,w}[\mathcal{O}_{X_z}], \tag{3.8}$$

**Corollary 3.9.** *Let $\lambda \in P^+$, $w \in W$ and $W_\lambda = \mathrm{Stab}(\lambda)$. Then, with notation as in (3.8),*

$$c^z_{\lambda,w} = \sum_{\substack{p\in\mathcal{T}^\lambda \\ wW_\lambda \geq \iota(p)\geq\phi(p)=zW_\lambda}} e^{p(1)},$$

$$c^z_{w_0\lambda,w} = (-1)^{\ell(w)+\ell(z)} c^{ww_0}_{\lambda,zw_0}, \qquad \text{and} \qquad c^z_{-\lambda,w} = (-1)^{\ell(w)+\ell(z)} c^{ww_0}_{-w_0\lambda,zw_0}.$$

**Proposition 3.10.** *For $1 \leq i \leq n$,* $\quad [\mathcal{O}_{X_{w_0 s_i}}] = 1 - e^{w_0\omega_i}[X^{-\omega_i}]$.

*Proof.* We shall show that

$$X^{-\omega_i}[\mathcal{O}_{X_{w_0}}] = e^{-w_0\omega_i}([\mathcal{O}_{X_{w_0}}] - [\mathcal{O}_{X_{w_0 s_i}}]), \tag{3.11}$$

and the result will follow by solving for $[\mathcal{O}_{X_{s_i w_0}}]$. Let $\omega_j = -w_0\omega_i$. By Corollary 3.9,

$$c^z_{-\omega_i,w_0} = (-1)^{\ell(w_0)+\ell(z)} c^1_{\omega_j,zw_0} = (-1)^{\ell(w_0)+\ell(z)} \sum_{\substack{p\in\mathcal{T}^{\omega_j} \\ zw_0\geq\iota(p)\geq\phi(p)=1}} e^{p(1)}.$$

The straight line path to $\omega_j$, $p_{\omega_j}$, has $\iota_{zw_0}(p_{\omega_j}) = \phi_{zw_0}(\omega_j)$ and is the unique path in $\mathcal{T}^{\omega_j}$ which may have final direction 1. Suppose $\phi_{zw_0}(p_{\omega_j}) = 1$. Then, since $s_j$ is the only simple reflection which is not in $\mathrm{Stab}(\omega_j)$, it must be that $zw_0 \not\geq s_k$ for all $k \neq j$. Thus $zw_0 = 1$ or $zw_0 = s_j$ and so $c^z_{-\omega_i,w_0} \neq 0$ only if $z = w_0$ or $z = s_j w_0 = w_0 s_i$. Now (3.11) follows since $p_{\omega_j}$ has endpoint $\omega_j = -w_0\omega_i$. ∎

**Corollary 3.12.** *Let $c^z_{wv}$ be as in (3.8). Then, for $1 \leq i \leq n$, $c^w_{w_0 s_i,w} = -(e^{-(w\omega_i - w_0\omega_i)} - 1)$, and*

$$c^z_{w_0 s_i,w} = (-1)^{\ell(w)+\ell(z)+1} \sum_{\substack{p\in\mathcal{T}^{-w_0\omega_i} \\ zw_0\geq\iota(p)\geq\phi(p)=ww_0}} e^{w_0\omega_i + p(1)}, \qquad \text{for } z \neq w.$$

*Proof.* This follows from Proposition 3.10 and Corollary 3.9 and the fact that, in the case when $z = w$, there is a unique path $p$ with $ww_0 = \iota(p) = \phi(p) = ww_0$ and endpoint $p(1) = ww_0(-w_0\omega_i) = -w\omega_i$. ∎

## 4. Converting to $H_T^*(G/B)$

The *graded nil-Hecke algebra* is the algebra $H_{\rm gr}$ given by generators $t_1, \ldots, t_n$ and $x_\lambda$, $\lambda \in P$, with relations

$$t_i^2 = 0, \qquad \underbrace{t_i t_j t_i \cdots}_{m_{ij}\text{ factors}} = \underbrace{t_j t_i t_j \cdots}_{m_{ij}\text{ factors}}, \qquad x_{\lambda+\mu} = x_\lambda + x_\mu, \quad \text{and} \quad x_\lambda t_i = t_i x_{s_i\lambda} + \langle \lambda, \alpha_i^\vee \rangle. \tag{4.1}$$

The subalgebra of $H_{\rm gr}$ generated by the $x_\lambda$ is the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$, where $x_i = x_{\omega_i}$, and $W$ acts on $\mathbb{Z}[x_1, \ldots, x_n]$ by

$$w x_\lambda = x_{w\lambda} \qquad \text{and} \qquad w(fg) = (wf)(wg), \qquad \text{for } w \in W,\ \lambda \in P,\ f, g \in \mathbb{Z}[x_1, \ldots, x_n].$$

Then the last formula in (4.1) generalizes to

$$f t_i = t_i(s_i f) + \frac{f - s_i f}{\alpha_i}, \qquad \text{for } f \in \mathbb{Z}[x_1, \ldots, x_n].$$

Let $t_w = t_{i_1} \cdots t_{i_p}$ for a reduced word $w = s_{i_1} \cdots s_{i_p}$ and let $\mathbb{Z}W^*$ be the subalgebra of $H_{\rm gr}$ spanned by the $t_w$, $w \in W$. Then

$$\{x_1^{m_1} \cdots x_n^{m_n} t_w \mid w \in W,\ \ m_i \in \mathbb{Z}_{\geq 0}\} \qquad \text{and} \qquad \{t_w x_1^{m_1} \cdots x_n^{m_n} \mid w \in W,\ \ m_i \in \mathbb{Z}_{\geq 0}\}$$

are bases of $H_{\rm gr}$.

Let $S = \mathbb{Z}[y_1, \ldots, y_n]$ and extend coefficients to $S$ so that $H_{{\rm gr},S} = S \otimes_{\mathbb{Z}} H_{\rm gr}$ and $S[x_1, \ldots, x_n] = S \otimes_{\mathbb{Z}} \mathbb{Z}[x_1, \ldots, x_n]$ are $S$-algebras. Define $H_T^*(G/B)$ to be the $H_{{\rm gr},S}$ module

$$H_T^*(G/B) = S\text{-span}\{[X_w] \mid w \in W\}, \tag{4.2}$$

so that the $[X_w]$, $w \in W$, are an $S$-basis of $K_T(G/B)$, with $H_{{\rm gr},S}$-action given by

$$x_i[X_1] = y_i[X_1], \qquad \text{and} \qquad t_i[X_w] = \begin{cases} [X_{ws_i}], & \text{if } ws_i > w, \\ 0, & \text{if } ws_i < w, \end{cases} \tag{4.3}$$

Let $y$ be the $S$-algebra homomorphism given by

$$\begin{array}{rccc} y\colon & S[x_1, \ldots, x_n] & \longrightarrow & S \\ & x_i & \longmapsto & y_i \end{array}$$

so that $H_T^*(G/B) \cong H_{{\rm gr},S} \otimes_{S[x_1,\ldots,x_n]} y$ as $H_{{\rm gr},S}$-modules Then, using analogous methods to the $K_T(G/B)$ case proves the following theorem, which gives the ring structure of $H^*T(G/B)$ (see also the proof of [KR, Prop. 2.9] for the same argument with (non-nil) graded Hecke algebras).

**Theorem 4.4.** *The composite map*

$$\begin{array}{rccccccc} \Phi\colon & S[x_1, \ldots, x_n] & \longrightarrow & H_{{\rm gr},S} t_{w_0} & \hookrightarrow & H_{{\rm gr},S} & \longrightarrow & H_T^*(G/B) \\ & f & \longmapsto & f t_{w_0} & & h & \longmapsto & h[X_1] \end{array}$$

*is surjective with kernel*

$$\ker \Phi = \langle f - y(f) \mid f \in S[x_1, \ldots, x_n]^W \rangle,$$

*the ideal of the ring $S_[x_1, \dots, x_n]$ generated by the elements $f - y(f)$ for $f \in S[x_1, \dots, x_n]^W$. Hence*

$$H_T^*(G/B) \;\cong\; \frac{\mathbb{Z}[y_1, \dots, y_n, x_1, \dots, x_n]}{\langle f - y(f) \mid f \in S[x_1, \dots, x_n]^W \rangle}$$

*has the structure of a ring.*

As a vector space $H_{\mathrm{gr}} = \mathbb{Z}[x_1, \dots, x_n] \otimes \mathbb{Z}W_{\mathrm{gr}}$. Let $\widehat{H_{\mathrm{gr}}} = \mathbb{Q}[[x_1, \dots, x_n]] \otimes \mathbb{Q}W_{gr}$ with multiplication determined by the relations in (4.1). Then $\widehat{H_{\mathrm{gr}}}$ is a completion of $H_{\mathrm{gr}}$ (this simply allows us to write infinite sums) and the elements of $\widehat{H_{\mathrm{gr}}}$ given by

$$\mathrm{ch}(X^\lambda) = \sum_{r \geq 0} \tfrac{1}{r!}\, x_\lambda^r \qquad \text{and} \qquad \mathrm{ch}(T_i) = t_i \cdot \frac{x_{\alpha_i}}{1 - \mathrm{ch}(X^{\alpha_i})} \tag{4.5}$$

satisfy the relations of $\tilde{H}$ and thus ch extends to a ring homomorphism $\mathrm{ch}\colon \tilde{H} \longrightarrow \widehat{H_{\mathrm{gr}}}$. It is this fact that really makes possible the transfer from $K$-theory to cohomomology possible. Though is it not difficult to check that the elements in (3.5) satisfy the defining relations of $\tilde{H}$ it is helpful to realize that these formulas come from geometry. As explained in [PR2], the action of $T_i$ on $K_T(G/B)$ and the action of $t_i$ on $H_T^*(G/B)$ are, respectively, the push-pull operators $\pi_i^*(\pi_i)_!$ and $\pi_i^*(\pi_i)_*$, where if $P_i$ is a minimal parabolic subgroup of $G$ then $\pi_i\colon G/P_i \to G/B$ is the natural surjection. Then the first formula in (3.5) is the definition of the Chern character, and the second formula is the Grothedieck-Riemann-Roch theorem applied to the map $\pi_i$. The factor $\alpha_i/(1 - \mathrm{ch}(X^{\alpha_i}))$ is the Todd class of the bundle of tangents along the fibers of $\pi_i$ (see [Hz, page 91]).

Then $\widehat{H_T^*}(G/B)_{\mathbb{Q}} = \mathbb{Q}[[y_1, \dots, y_n]] \otimes_{\mathbb{Z}[y_1, \dots, y_n]} H_T^*(G/B)$ is the appropriate completion of $H_T^*(G/B)$ to use to transfer the ring homomorphism $\mathrm{ch}\colon \tilde{H}_R \to \widehat{H_{\mathrm{gr}}}$ to a ring homomorphism

$$\mathrm{ch}\colon K_T(G/B) \longrightarrow \widehat{H_T^*}(G/B)_{\mathbb{Q}} \qquad \text{by setting} \quad \mathrm{ch}(h[\mathcal{O}_{X_1}]) = \mathrm{ch}(h)[X_1], \quad \text{for } h \in \tilde{H}_R. \tag{4.6}$$

The ring $\widehat{H_T^*}(G/B)_{\mathbb{Q}}$ is a graded ring with

$$\deg(y_i) = 1 \qquad \text{and} \qquad \deg([X_w]) = \ell(w_0) - \ell(w), \tag{4.7}$$

$$\text{and, } \quad \text{for } w \in W, \qquad \mathrm{ch}([\mathcal{O}_{X_w}]) = [X_w] + \text{ higher degree terms.} \tag{4.8}$$

In summary, if $e_i = e^{\omega_i}$, $X_i = X^{\omega_i}$, $y_i = y_{\omega_i}$, $x_i = x_{\omega_i}$,

$$\begin{matrix} R[X] = \mathbb{Z}[e_1^{\pm 1}, \dots, e_n^{\pm 1}, X_1^{\pm 1}, \dots, X_n^{\pm 1}], \\ \mathbb{Z}[X] = \mathbb{Z}[X_1^{\pm 1}, \dots, X_n^{\pm 1}], \end{matrix} \qquad \text{and} \quad \widehat{S}[x_1, \dots, x_n] = \mathbb{Q}[[y_1, \dots, y_n]][x_1, \dots, x_n],$$

then there is a commutative diagram of ring homomorphisms

$$\begin{array}{ccc} K_T(G/B) = \dfrac{R[X]}{\langle f - e(f) \mid f \in R[X]^W \rangle} & \xrightarrow{\mathrm{ch}} & H_T^*(G/B)_{\mathbb{Q}} = \dfrac{\widehat{S}[x_1, \dots, x_n]}{\langle f - y(f) \mid f \in \widehat{S}[x_1, \dots, x_n]^W \rangle} \\ \big\downarrow{\scriptstyle e_i = 1} & & \big\downarrow{\scriptstyle y_i = 0} \\ K(G/B) = \dfrac{\mathbb{Z}[X]}{\langle f - f(1) \mid f \in \mathbb{Z}[X]^W \rangle} & \xrightarrow{\mathrm{ch}} & H^*(G/B)_{\mathbb{Q}} = \dfrac{\mathbb{Q}[x_1, \dots, x_n]}{\langle f - f(0) \mid f \in \mathbb{Q}[x_1, \dots, x_n]^W \rangle}. \end{array}$$

## 5. Rank two and a positivity conjecture

In this section we will give explicit formulas for the rank two root systems. The data supports the following positivity conjecture which generalizes the theorems of Brion [Br, formula before Theorem 1] and Graham [Gr, Corollary 4.1].

**Conjecture 5.1.** *For $\beta \in R^+$ let $y_\beta = e^{-\beta}$ and $a_\beta = e^{-\beta} - 1$ and let $d(w) = \ell(w_0) - \ell(w)$ for $w \in W$. Let $c_{wv}^z$ be the structure constants of $K_T(G/B)$ with respect to the basis $\{[\mathcal{O}_{X_w}] \mid w \in W\}$ as defined in (0.1). Then*

$$c_{wv}^z = (-1)^{d(w)+d(v)-d(z)} f(\alpha, y), \qquad \text{where} \quad f(\alpha, y) \in \mathbb{Z}_{\geq 0}[\alpha_\beta, y_\beta \mid \beta \in R^+],$$

*that is, $f(\alpha, y)$ is a polynomial in the variables $\alpha_\beta$ and $y_\beta$, $\beta \in R^+$, which has nonnegative integral coefficients.*

In the following, for brevity, use the following notations:

$$\begin{array}{llllll}
\text{in } K_T(G/B), & [w] = [\mathcal{O}_{X_w}], & \alpha_{rs} = e^{-(r\alpha_1+s\alpha_2)} - 1, & \text{and} & y_{rs} = e^{-(r\alpha_1+s\alpha_2)}, \\
\text{in } K(G/B), & [w] = [\mathcal{O}_{X_w}], & \alpha_{rs} = 0, & \text{and} & y_{rs} = 1, \\
\text{in } H_T^*(G/B), & [w] = [X_w], & \alpha_{rs} = r\alpha_1 + s\alpha_2, & \text{and} & y_{rs} = 1, \\
\text{in } H^*(G/B), & [w] = [X_w], & \alpha_{rs} = 0, & \text{and} & y_{rs} = 1,
\end{array}$$

and in $H_T^*(G/B)$ and in $H^*(G/B)$ the terms in $\{\}$ brackets do not appear.

**Type $A_2$.** For the root system $R$ of type $A_2$

$$\begin{array}{llll}
\alpha_1 = -\omega_1 + 2\omega_2, & \lambda_1 = \rho, & \lambda_{s_1} = \omega_2 = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2, & \lambda_{s_2s_1} = s_2\omega_2 = \frac{1}{3}\alpha_1 - \frac{1}{3}\alpha_2, \\
\alpha_2 = \phantom{-}2\omega_1 - \omega_2, & \lambda_{w_0} = 0, & \lambda_{s_2} = \omega_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2, & \lambda_{s_1s_2} = s_1\omega_1 = -\frac{1}{3}\alpha_1 + \frac{1}{3}\alpha_2.
\end{array}$$

Formulas for the Schubert classes in terms of homogeneous line bundles can be given by

$$\begin{array}{ll}
[s_1s_2s_1] = 1, & [1] = (1 - e^{s_1\omega_1}X^{-\omega_1})[s_1] = (1 - e^{s_2\omega_2}X^{-\omega_2})[s_2], \\
[s_2s_1] = 1 - e^{-\omega_1}X^{-\omega_2}, & [s_1s_2] = 1 - e^{-\omega_2}X^{-\omega_1} \\
[s_1] = (1 - e^{s_2\omega_2}X^{-\omega_2})[s_2s_1], & [s_2] = (1 - e^{s_1\omega_1}X^{-\omega_1})[s_1s_2],
\end{array}$$

and

$$\begin{gathered}
[s_1s_2s_1] = 1, \qquad [s_1s_2] = 1 - e^{-\omega_2}X^{-\omega_1}, \qquad [s_2s_1] = 1 - e^{-\omega_1}X^{-\omega_2}, \\
[s_1] = 1 - e^{-\omega_2}X^{-s_1\omega_1} - e^{-\omega_2}X^{-\omega_1} + e^{-2\omega_2}X^{-\omega_2}, \\
[s_2] = 1 - e^{-\omega_1}X^{-s_2\omega_2} - e^{-\omega_1}X^{-\omega_2} + e^{-2\omega_1}X^{-\omega_1}, \\
[1] = 1 - e^{-\omega_2}X^{-s_1\omega_1} - e^{-\omega_1}X^{-s_2\omega_2} + e^{-2\omega_1}X^{-\omega_1} + e^{-2\omega_2}X^{-\omega_2} - e^{-\rho}X^{-\rho}.
\end{gathered}$$

The multiplication of the Schubert classes is given by

$$\begin{array}{rrr}
[1]^2 = -\alpha_{10}\alpha_{01}\alpha_{11}[1], & [s_1]^2 = \alpha_{01}\alpha_{11}[s_1], & [s_2]^2 = \alpha_{01}\alpha_{11}[s_2], \\
[1][s_1] = \alpha_{01}\alpha_{11}[1], & [s_1][s_2] = -\alpha_{11}[1], & [s_2][s_1s_2] = -\alpha_{11}[s_2], \\
[1][s_2] = \alpha_{10}\alpha_{11}[1], & [s_1][s_1s_2] = y_{01}[1] - \alpha_{01}[s_1], & [s_2][s_2s_1] = y_{10}[1] - \alpha_{10}[s_2], \\
[1][s_1s_2] = -\alpha_{11}[1], & [s_1][s_2s_1] = -\alpha_{11}[s_1], & \\
[1][s_2s_1] = -\alpha_{11}[1], & &
\end{array}$$

$$[s_1s_2]^2 = y_{01}[s_2] - \alpha_{01}[s_1s_2], \qquad [s_2s_1]^2 = y_{10}[s_1] - \alpha_{10}[s_2s_1].$$
$$[s_1s_2][s_2s_1] = \{\, -[1]\,\} + [s_1] + [s_2],$$

**Type $B_2$.** For the root system $R$ of type $B_2$

$$\begin{aligned} \alpha_1 &= \phantom{-}2\omega_1 - \omega_2, & \lambda_1 &= \rho = 2\alpha_1 + \tfrac{3}{2}\alpha_2, & \lambda_{s_1} &= \omega_2 = \alpha_1 + \alpha_2, \\ \alpha_2 &= -2\omega_1 + 2\omega_2, & \lambda_{w_0} &= 0, & \lambda_{s_2} &= \omega_1 = \alpha_1 + \tfrac{1}{2}\alpha_2, \end{aligned}$$

$$\begin{aligned} \lambda_{s_2s_1} &= s_2\omega_2 = \alpha_1, & \lambda_{s_1s_2s_1} &= s_1s_2\omega_2 = -\alpha_1, \\ \lambda_{s_1s_2} &= s_1\omega_1 = \tfrac{1}{2}\alpha_2, & \lambda_{s_2s_1s_2} &= s_2s_1\omega_1 = -\tfrac{1}{2}\alpha_2. \end{aligned}$$

Formulas for the Schubert classes in terms of homogeneous line bundles can be given by

$$\begin{aligned} &[s_1s_2s_1s_2] = 1, && [1] = (1 - e^{s_1\omega_1}X^{-\omega_1})[s_1] = (1 - e^{s_2\omega_2}X^{-\omega_2})[s_2], \\ &[s_1s_2s_1] = 1 - e^{-\omega_2}X^{-\omega_2}, && [s_2s_1s_2] = 1 - e^{-\omega_1}X^{-\omega_1}, \\ &[s_2s_1] = (1 - e^{-\omega_1}X^{-s_1\omega_1})[s_2s_1s_2], && [s_1s_2] = (1 - e^{s_2s_1\omega_1}X^{-\omega_1})[s_2s_1s_2], \\ &[s_1] = (1 - e^{s_2\omega_2}X^{-\omega_2})[s_2s_1], && [s_2] = (1 - e^{s_1\omega_1}X^{-\omega_1})[s_1s_2], \end{aligned}$$

and

$$[s_1s_2s_1s_2] = 1, \qquad [s_1s_2s_1] = 1 - e^{-\omega_2}X^{-\omega_2}, \qquad [s_2s_1s_2] = 1 - e^{-\omega_1}X^{-\omega_1},$$
$$\begin{aligned} [s_1s_2] &= (1 - e^{-\omega_2}) - e^{-\omega_2}X^{-\omega_2} - e^{-\omega_2}X^{-s_2\omega_2} + (e^{-\rho} + e^{-s_1\rho})X^{-\omega_1}, \\ [s_2s_1] &= 1 - e^{-\omega_1}X^{-\omega_1} - e^{-\omega_1}X^{-s_1\omega_1} + e^{-2\omega_1}X^{-\omega_2}, \\ [s_1] &= (1 - e^{-\omega_2}) + (e^{-\rho} + e^{-s_1\rho})X^{-s_1\omega_1} + (e^{-\rho} + e^{-s_1\rho})X^{-\omega_1} \\ &\qquad - e^{-\omega_2}X^{-s_1s_2\omega_2} - e^{-\omega_2}X^{-s_2\omega_2} - (e^{-2\omega_2} + e^{-\omega_2})X^{-\omega_2}, \\ [s_2] &= (1 + e^{-2\omega_1}) + e^{-2\omega_1}X^{-s_2\omega_2} + e^{-2\omega_1}X^{-\omega_2} \\ &\qquad - e^{-\omega_1}X^{-s_2s_1\omega_1} - e^{-\omega_1}X^{-s_1\omega_1} - (e^{-3\omega_1} + e^{-\omega_1})X^{-\omega_1}, \\ [1] &= (1 + e^{-2\omega_1}) - e^{-\omega_1}X^{-s_2s_1\omega_1} + (e^{-\rho} + e^{-s_1\rho})X^{-s_1\omega_1} - (e^{-3\omega_1} + e^{-\omega_1})X^{-\omega_1} \\ &\qquad - e^{-\omega_2}X^{-s_1s_2\omega_2} + e^{-2\omega_1}X^{-s_2\omega_2} - (e^{-2\omega_2} + e^{-\omega_2})X^{-\omega_2} + e^{-\rho}X^{-\rho}. \end{aligned}$$

The multiplication of the Schubert classes is given by

$$\begin{aligned} [1]^2 &= \alpha_{10}\alpha_{01}\alpha_{11}\alpha_{21}[1], \\ [1][s_1] &= -\alpha_{01}\alpha_{11}\alpha_{21}[1], \\ [1][s_2] &= -\alpha_{10}\alpha_{11}\alpha_{21}[1], \\ [1][s_1s_2] &= \alpha_{11}\alpha_{21}[1], \\ [1][s_2s_1] &= \alpha_{11}\alpha_{21}[1], \\ [1][s_1s_2s_1] &= -\alpha_{11}(1 + y_{11})[1], \\ [1][s_2s_1s_2] &= -\alpha_{21}[1], \end{aligned}$$

$$\begin{aligned} [s_1s_2s_1]^2 &= \{\, -y_{11}[s_1]\,\} + (y_{01} + y_{11})[s_2s_1] - \alpha_{01}[s_1s_2s_1], \\ [s_1s_2s_1][s_2s_1s_2] &= \{\, [1] - [s_1] - [s_2]\,\} + [s_1s_2] + [s_2s_1], \end{aligned}$$

$$\begin{aligned} [s_2s_1s_2]^2 &= y_{10}[s_1s_2] - \alpha_{10}[s_2s_1s_2], \\ [s_2s_1]^2 &= -\alpha_{21}y_{10}[s_1] + \alpha_{10}\alpha_{21}[s_2s_1], \end{aligned}$$

$$\begin{aligned} [s_2s_1][s_1s_2s_1] &= y_{21}[s_1] - \alpha_{21}[s_2s_1], \\ [s_2s_1][s_2s_1s_2] &= \{\, -y_{10}[1]\,\} + y_{10}[s_1] + y_{10}[s_2] - \alpha_{10}[s_2s_1], \end{aligned}$$

$$\begin{aligned} [s_1]^2 &= -\alpha_{01}\alpha_{11}\alpha_{21}[s_1], \\ [s_1][s_2] &= \alpha_{11}\alpha_{21}[1], \\ [s_1][s_1s_2] &= -\alpha_{11}(y_{01} + y_{11})[1] + \alpha_{01}\alpha_{11}[s_1], \\ [s_1][s_2s_1] &= \alpha_{11}\alpha_{21}[s_1], \\ [s_1][s_1s_2s_1] &= -\alpha_{11}(1 + y_{11})[s_1], \\ [s_1][s_2s_1s_2] &= y_{11}[1] - \alpha_{11}[s_1], \end{aligned}$$

$$\begin{aligned} [s_2]^2 &= -\alpha_{10}\alpha_{11}\alpha_{21}[s_2], \\ [s_2][s_1s_2] &= \alpha_{11}\alpha_{21}[s_2], \\ [s_2][s_2s_1] &= -\alpha_{21}y_{10}[1] + \alpha_{10}\alpha_{21}[s_2], \\ [s_2][s_1s_2s_1] &= y_{21}[1] - \alpha_{21}[s_2], \\ [s_2][s_2s_1s_2] &= -\alpha_{21}[s_2], \end{aligned}$$

$$\begin{aligned}
[s_1s_2]^2 &= -\alpha_{11}(y_{01}+y_{11})[s_2] + \alpha_{01}\alpha_{11}[s_1s_2],\\
[s_1s_2][s_2s_1] &= (\{\,\alpha_{11}\,\} + y_{21})[1] - \alpha_{11}[s_1] - \alpha_{21}[s_2],\\
[s_1s_2][s_1s_2s_1] &= \{\,-(y_{01}+y_{11})[1]\,\} + y_{01}[s_1] + (y_{11}+y_{12})[s_2] - \alpha_{01}[s_1s_2],\\
[s_1s_2][s_2s_1s_2] &= y_{11}[s_2] - \alpha_{11}[s_1s_2],
\end{aligned}$$

$$\begin{aligned}
[s_2s_1]^2 &= -\alpha_{21}y_{10}[s_1] + \alpha_{10}\alpha_{21}[s_2s_1],\\
[s_2s_1][s_1s_2s_1] &= y_{21}[s_1] - \alpha_{21}[s_2s_1],\\
[s_2s_1][s_2s_1s_2] &= \{\,-y_{10}[1]\,\} + y_{10}[s_1] + y_{10}[s_2] - \alpha_{10}[s_2s_1],
\end{aligned}$$

**Type $G_2$.** For the root system $R$ of type $G_2$

$$\begin{aligned}
&\lambda_1 = \rho = 5\alpha + 3\alpha_2, && \lambda_{s_1s_2s_1} = s_1s_2\omega_2 = \alpha_2,\\
&\lambda_{s_1} = \omega_2 = 3\alpha_1 + 2\alpha_2, && \lambda_{s_2s_1s_2s_1} = s_2s_1s_2\omega_2 = -\alpha_2,\\
&\lambda_{s_2} = \omega_1 = 2\alpha_1 + \alpha_2, && \lambda_{s_1s_2s_1s_2} = s_1s_2s_1\omega_1 = -\alpha_1,\\
&\lambda_{s_2s_1} = s_2\omega_2 = 3\alpha_1 + \alpha_2, && \lambda_{s_1s_2s_1s_2s_1} = s_1s_2s_1s_2\omega_2 = -3\alpha_1 - \alpha_2,\\
&\lambda_{s_1s_2} = s_1\omega_1 = \alpha_1 + \alpha_2, && \lambda_{s_2s_1s_2s_1s_2} = s_2s_1s_2s_1\omega_1 = -\alpha_1 - \alpha_2,\\
&\lambda_{s_2s_1s_2} = s_2s_1\omega_1 = \alpha_1, && \lambda_{w_0} = 0.
\end{aligned}$$

Formulas for the Schubert classes in terms of homogeneous line bundles can be given by

$$\begin{aligned}
&[s_1s_2s_1s_2s_1s_2] = 1, && [1] = (1 - e^{s_1\omega_1}X^{-\omega_1})[s_1] = (1 - e^{s_2\omega_2}X^{-\omega_2})[s_2],\\
&[s_1s_2s_1s_2s_1] = 1 - e^{-\omega_2}X^{-\omega_2}, && [s_2s_1s_2s_1s_2] = 1 - e^{-\omega_1}X^{-\omega_1},\\
&[s_2s_1s_2s_1] = (1 - e^{-\omega_1}X^{-s_1\omega_1})[s_2s_1s_2s_1s_2], && [s_1s_2s_1s_2] = (1 - e^{-s_1\omega_1}X^{-\omega_1})[s_2s_1s_2s_1s_2],\\
&[s_1s_2s_1] = \text{see below}, && [s_2s_1s_2] = \frac{1 - e^{-s_2s_2\omega_1}X^{-\omega_1}}{1 + X^{-\omega_1}}[s_1s_2s_1s_2],\\
&[s_2s_1] = (1 - e^{-\omega_1}X^{-s_1s_2s_1\omega_1})[s_2s_1s_2], && [s_1s_2] = (1 - e^{s_2s_1\omega_1}X^{-\omega_1})[s_1s_2],\\
&[s_1] = (1 - e^{s_2\omega_2}X^{-\omega_2})[s_2s_1], && [s_2] = (1 - e^{s_1\omega_1}X^{-\omega_1})[s_1s_2],
\end{aligned}$$

$$[s_1s_2s_1] = \frac{(1 - e^{-\alpha_2}X^{-\omega_2})[s_2s_1s_2s_1] + e^{-\alpha_2}(1 + e^{\omega_1}X^{-\omega_2})[s_2s_1]}{1 + e^{-\alpha_2}},$$

and

$$
\begin{aligned}
[w_0] &= 1, \qquad [s_2s_1s_2s_1s_2] = 1 - y_{21}X^{-\omega_1}, \qquad [s_1s_2s_1s_2s_1] = 1 - y_{32}X^{-\omega_2}, \\
[s_2s_1s_2s_1] &= 1 - y_{21}X^{-\omega_1} - y_{21}X^{-s_1\omega_1} + y_{42}X^{-\omega_2}, \\
[s_1s_2s_1s_2] &= (1-y_{32}) + (y_{22}+y_{42}+y_{43}+y_{53})X^{-\omega_1} - y_{32}X^{-s_1\omega_1} - y_{32}X^{-s_2s_1\omega_1} \\
&\qquad - y_{32}X^{-\omega_2} - y_{32}X^{-s_2\omega_2}, \\
[s_2s_1s_2] &= (1-y_{21}+y_{42}) + (y_{42}-y_{21}-y_{52}-y_{53}-y_{63})X^{-\omega_1} + (y_{42}-y_{21})X^{-s_1\omega_1} \\
&\qquad + (y_{42}-y_{21})X^{-s_2s_1\omega_1} + y_{42}X^{-\omega_2} + y_{42}X^{-s_2\omega_2}, \\
[s_1s_2s_1] &= (1-2y_{32}) + (y_{22}+y_{42}+y_{43}+y_{53})X^{-\omega_1} + (y_{22}+y_{42}+y_{43}+y_{53})X^{-s_1\omega_1} \\
&\qquad - y_{32}X^{-s_2s_1\omega_1} - y_{32}X^{-s_1s_2s_1\omega_1} \\
&\qquad - (y_{32}+y_{43}+y_{53})X^{-\omega_2} - y_{32}X^{-s_2\omega_2} - y_{32}X^{-s_1s_2\omega_2}, \\
[s_2s_1] &= (1-y_{21}+2y_{42}) + (y_{42}-y_{21}-y_{52}-y_{53}-y_{63})X^{-\omega_1} \\
&\qquad + (y_{42}-y_{21}-y_{32}-y_{53}-y_{63})X^{-s_1\omega_1} + (y_{42}-y_{21})X^{-s_2s_1\omega_1} \\
&\qquad + (y_{42}-y_{21})X^{-s_1s_2s_1\omega_1} + (y_{42}+y_{63})X^{-\omega_2} + y_{42}X^{-s_2\omega_2} + y_{42}X^{-s_1s_2\omega_2}, \\
[s_1s_2] &= 1 - y_{11} - y_{21} - y_{32} - y_{43} - y_{53} + (y_{22}+y_{32})(1+y_{10}+y_{20})X^{-\omega_1} \\
&\qquad + (y_{22}+y_{32}+y_{42})X^{-s_1\omega_1} + (y_{22}+y_{32}+y_{42})X^{-s_2s_1\omega_1} \\
&\qquad - (y_{32}+y_{43}+y_{53})X^{-\omega_2} - (y_{32}+y_{43}+y_{53})X^{-s_2\omega_2} - y_{32}X^{-s_1s_2\omega_2} - y_{32}X^{-s_2s_1s_2\omega_2}, \\
[s_2] &= (1+y_{31}+y_{32}+2y_{42}+y_{63}) - (y_{21}+y_{52}+y_{53}+y_{84})X^{-\omega_1} - (y_{21}+y_{52}+y_{53})X^{-s_1\omega_1} \\
&\qquad - (y_{21}+y_{52}+y_{53})X^{-s_2s_1\omega_1} - y_{21}X^{-s_1s_2s_1\omega_1} - y_{21}X^{-s_2s_1s_2s_1\omega_1} \\
&\qquad + (y_{42}+y_{63})X^{-\omega_2} + (y_{42}+y_{63})X^{-s_2\omega_2} + y_{42}X^{-s_1s_2\omega_2} + y_{42}X^{-s_2s_1s_2\omega_2}, \\
[s_1] &= 1 - (y_{11}+y_{21}+y_{32}+2y_{43}+2y_{53}) + (y_{22}+y_{54})(1+y_{10}+y_{20})X^{-\omega_1} \\
&\qquad + (y_{22}+y_{54})(1+y_{10}+y_{20})X^{-s_1\omega_1} + (y_{22}+y_{32}+y_{42})X^{-s_2s_1\omega_1} \\
&\qquad + (y_{22}+y_{32}+y_{42})X^{-s_1s_2s_1\omega_1} - (y_{32}+y_{43}+y_{53}+y_{64})X^{-\omega_2} - (y_{32}+y_{43}+y_{53})X^{-s_2\omega_2} \\
&\qquad - (y_{32}+y_{43}+y_{53})X^{-s_1s_2\omega_2} - y_{32}X^{-s_2s_1s_2\omega_2} - y_{32}X^{-s_1s_2s_1s_2\omega_2}, \\
[1] &= (1+y_{31}+y_{42}+y_{63}-y_{53}-y_{43}) - y_{21}(1+y_{32})^2X^{-\omega_1} \\
&\qquad + y_{22}(1+y_{10}+y_{20})(1+y_{21}+y_{31})X^{-s_1\omega_1} - (y_{21}+y_{52}+y_{53})X^{-s_2s_1\omega_1} \\
&\qquad + y_{22}X^{-s_1s_2s_1\omega_1} - y_{21}X^{-s_2s_1s_2s_1\omega_1} - y_{32}(1+y_{11})(1+y_{21})X^{-\omega_2} + (y_{42}+y_{63})X^{-s_2\omega_2} \\
&\qquad - (y_{32}+y_{43}+y_{53})X^{-s_1s_2\omega_2} + y_{42}X^{-s_2s_1s_2\omega_2} - y_{32}X^{-s_1s_2s_1s_2\omega_2} + y_{53}X^{-\rho}.
\end{aligned}
$$

The multiplication of the Schubert classes is given by

$$
\begin{aligned}
[1]^2 &= \alpha_{10}\alpha_{01}\alpha_{11}\alpha_{21}\alpha_{31}\alpha_{32}[1], \\
[1][s_1] &= -\alpha_{01}\alpha_{11}\alpha_{21}\alpha_{31}\alpha_{32}[1], \\
[1][s_2] &= -\alpha_{10}\alpha_{11}\alpha_{21}\alpha_{31}\alpha_{32}[1], \\
[1][s_1s_2] &= \alpha_{11}\alpha_{21}\alpha_{31}\alpha_{32}[1], \\
[1][s_2s_1] &= \alpha_{11}\alpha_{21}\alpha_{31}\alpha_{32}[1], \\
[1][s_1s_2s_1] &= -\alpha_{11}\alpha_{21}\alpha_{32}(1+y_{11}+y_{21})[1],
\end{aligned}
\qquad
\begin{aligned}
[1][s_2s_1s_2] &= -\alpha_{21}\alpha_{31}\alpha_{32}[1], \\
[1][s_1s_2s_1s_2] &= \alpha_{21}\alpha_{32}(1+y_{21})[1], \\
[1][s_2s_1s_2s_1] &= \alpha_{21}\alpha_{32}(1+y_{21})[1], \\
[1][s_1s_2s_1s_2s_1] &= -\alpha_{32}(1+y_{32})[1], \\
[1][s_2s_1s_2s_1s_2] &= -\alpha_{21}(1+y_{21})[1],
\end{aligned}
$$

$$
\begin{aligned}
[s_1]^2 &= -\alpha_{01}\alpha_{11}\alpha_{21}\alpha_{31}\alpha_{32}[s_1]\\
[s_1][s_2] &= \alpha_{11}\alpha_{21}\alpha_{31}\alpha_{32}[1]\\
[s_1][s_1s_2] &= -\alpha_{11}\alpha_{21}\alpha_{32}(y_{01}+y_{11}+y_{21})[1]+\alpha_{01}\alpha_{11}\alpha_{21}\alpha_{32}[s_1]\\
[s_1][s_2s_1] &= \alpha_{11}\alpha_{21}\alpha_{31}\alpha_{32}[s_1]\\
[s_1][s_1s_2s_1] &= -\alpha_{11}\alpha_{21}\alpha_{32}(1+y_{11}+y_{21})[s_1]\\
[s_1][s_2s_1s_2] &= \alpha_{21}\alpha_{32}(y_{11}+y_{21})[1]-\alpha_{11}\alpha_{21}\alpha_{32}[s_1]\\
[s_1][s_1s_2s_1s_2] &= -\alpha_{32}(y_{22}+y_{32})[1]+\alpha_{11}\alpha_{32}(1+y_{11})[s_1]\\
[s_1][s_2s_1s_2s_1] &= \alpha_{21}\alpha_{32}(1+y_{21})[s_1]\\
[s_1][s_1s_2s_1s_2s_1] &= -\alpha_{32}(1+y_{32})[s_1]\\
[s_1][s_2s_1s_2s_1s_2] &= y_{32}[1]-\alpha_{32}[s_1]
\end{aligned}
$$

$$
\begin{aligned}
[s_2]^2 &= -\alpha_{10}\alpha_{11}\alpha_{21}\alpha_{31}\alpha_{32}[s_2]\\
[s_2][s_1s_2] &= \alpha_{11}\alpha_{21}\alpha_{31}\alpha_{32}[s_2]\\
[s_2][s_2s_1] &= -\alpha_{21}\alpha_{31}\alpha_{32}y_{10}[1]+\alpha_{10}\alpha_{21}\alpha_{31}\alpha_{32}[s_2]\\
[s_2][s_1s_2s_1] &= \alpha_{21}\alpha_{32}(y_{21}+y_{31})[1]-\alpha_{21}\alpha_{31}\alpha_{32}[s_2]\\
[s_2][s_2s_1s_2] &= -\alpha_{21}\alpha_{31}\alpha_{32}[s_2]\\
[s_2][s_1s_2s_1s_2] &= \alpha_{21}\alpha_{32}(1+y_{21})[s_2]\\
[s_2][s_2s_1s_2s_1] &= -\alpha_{21}(y_{31}+y_{52})[1]+\alpha_{21}\alpha_{31}(1+y_{21})[s_2]\\
[s_2][s_1s_2s_1s_2s_1] &= y_{63}[1]-\alpha_{21}(1+y_{21}+y_{42})[s_2]\\
[s_2][s_2s_1s_2s_1s_2] &= -\alpha_{21}(1+y_{21})[s_2]
\end{aligned}
$$

$$
\begin{aligned}
[s_1s_2]^2 &= -\alpha_{11}\alpha_{21}\alpha_{32}(y_{01}+y_{11}+y_{21})[s_2]+\alpha_{01}\alpha_{11}\alpha_{21}\alpha_{32}[s_1s_2]\\
[s_1s_2][s_2s_1] &= \alpha_{21}\alpha_{32}(y_{11}+y_{21}+\alpha_{31})[1]-\alpha_{11}\alpha_{21}\alpha_{32}[s_1]-\alpha_{21}\alpha_{31}\alpha_{32}[s_2]\\
[s_1s_2][s_1s_2s_1] &= -\alpha_{32}(y_{32}+y_{42}\{\,+\alpha_{11}(y_{01}+2y_{11}+y_{21})\,\})[1]+\alpha_{11}\alpha_{32}(y_{01}+y_{11})[s_1]\\
&\quad+\big(\alpha_{31}\alpha_{32}y_{11}+\alpha_{11}\alpha_{32}(y_{01}+y_{11}+y_{21})\big)[s_2]-\alpha_{01}\alpha_{11}\alpha_{32}[s_1s_2]\\
[s_1s_2][s_2s_1s_2] &= \alpha_{21}\alpha_{32}(y_{11}+y_{21})[s_2]-\alpha_{11}\alpha_{21}\alpha_{32}[s_1s_2]\\
[s_1s_2][s_1s_2s_1s_2] &= -\alpha_{32}(y_{22}+y_{32})[s_2]+\alpha_{11}\alpha_{32}(1+y_{11})[s_1s_2]\\
[s_1s_2][s_2s_1s_2s_1] &= \big(y_{63}\,\{+\alpha_{32}(y_{11}+y_{21})\,\}\big)[1]-\alpha_{32}y_{11}[s_1]-\big(\alpha_{32}(y_{11}+y_{21})+\alpha_{31}y_{32}\big)[s_2]\\
&\quad+\alpha_{11}\alpha_{32}[s_1s_2]\\
[s_1s_2][s_1s_2s_1s_2s_1] &= \{\,-(y_{33}+y_{43}+y_{53})[1]\,\}+y_{33}[s_1]+(y_{33}+y_{43}+y_{53})[s_2]\\
&\quad-\alpha_{11}(1+y_{11}+y_{22})[s_1s_2]\\
[s_1s_2][s_2s_1s_2s_1s_2] &= y_{32}[s_2]-\alpha_{32}[s_1s_2]
\end{aligned}
$$

$$
\begin{aligned}
[s_2s_1]^2 &= -\alpha_{21}\alpha_{31}\alpha_{32}y_{10}[s_1] + \alpha_{10}\alpha_{21}\alpha_{31}\alpha_{32}[s_2s_1] \\
[s_2s_1][s_1s_2s_1] &= \alpha_{21}\alpha_{31}(y_{21}+y_{31})[s_1] - \alpha_{21}\alpha_{31}\alpha_{32}[s_2s_1] \\
[s_2s_1][s_2s_1s_2] &= -\alpha_{21}(y_{51}+y_{52}\{\,+\alpha_{31}y_{10}\,\})[1] + \alpha_{21}(\alpha_{10}y_{31}+\alpha_{32}y_{10})[s_1] \\
&\quad + \alpha_{21}\alpha_{31}(y_{10}+y_{21})[s_2] - \alpha_{10}\alpha_{21}\alpha_{31}[s_2s_1] \\
[s_2s_1][s_1s_2s_1s_2] &= \big(y_{62}\{\,+\alpha_{31}(y_{21}+y_{31})\,\}\big)[1] - \big(\alpha_{31}y_{21}+\alpha_{10}(y_{31}+y_{41})\big)[s_1] \\
&\quad - \big(\alpha_{31}y_{21}+\alpha_{32}y_{31}\big)[s_2] + \alpha_{21}\alpha_{31}[s_2s_1] \\
[s_2s_1][s_2s_1s_2s_1] &= -\alpha_{21}(y_{31}+y_{52})[s_1] + \alpha_{21}\alpha_{31}(1+y_{21})[s_2s_1] \\
[s_2s_1][s_1s_2s_1s_2s_1] &= y_{63}[s_1] - \alpha_{21}(1+y_{21}+y_{42})[s_2s_1] \\
[s_2s_1][s_2s_1s_2s_1s_2] &= \{\,-y_{31}[1]\,\} + y_{31}[s_1] + y_{31}[s_2] - \alpha_{31}[s_2s_1]
\end{aligned}
$$

$$
\begin{aligned}
[s_1s_2s_1]^2 &= -\alpha_{32}(y_{32}+y_{42}\{\,+\alpha_{11}(y_{11}+y_{21})\,\})[s_1] \\
&\quad + \big(\alpha_{11}\alpha_{32}(y_{01}+y_{11}+y_{21})+\alpha_{31}\alpha_{32}y_{11}\big)[s_2s_1] - \alpha_{01}\alpha_{11}\alpha_{32}[s_1s_2s_1] \\
[s_1s_2s_1][s_2s_1s_2] &= \big(1\{\,+\alpha_{11}(y_{11}+y_{22}+y_{33}+y_{31}+y_{42})+\alpha_{31}(y_{21}+y_{32})+\alpha_{32}y_{21}\,\}\big)[1] \\
&\quad - \big(\alpha_{11}(y_{21}+\alpha_{32})+\alpha_{10}(y_{31}+y_{41}+y_{32}+y_{42})\big)[s_1] \\
&\quad - (\alpha_{31}(y_{21}+y_{32})+\alpha_{11}(y_{21}+y_{32}+y_{31}+\alpha_{42})[s_2] \\
&\quad + \alpha_{11}\alpha_{32}[s_1s_2] + \alpha_{21}\alpha_{31}[s_2s_1] \\
[s_1s_2s_1][s_1s_2s_1s_2] &= \{\,-(y_{33}+2y_{43}+y_{53}+\alpha_{11}(y_{01}+y_{11})+\alpha_{21}(y_{11}+y_{21}))[1]\,\} \\
&\quad + \big(y_{33}+y_{43}\{\,+\alpha_{11}(y_{01}+y_{11})+\alpha_{21}(y_{11}+y_{21})\,\}\big)[s_1] \\
&\quad \big((y_{33}+y_{43}+y_{53})\{\,+\alpha_{11}(y_{01}+y_{11})+\alpha_{21}(y_{11}+y_{21})\,\}\big)[s_2] \\
&\quad - \alpha_{11}(y_{01}+y_{11}+y_{22})[s_1s_2] - \big(\alpha_{11}(y_{01}+y_{11})+\alpha_{21}(y_{11}+y_{21})\big)[s_2s_1] \\
&\quad + \alpha_{01}\alpha_{11}[s_1s_2s_1] \\
[s_1s_2s_1][s_2s_1s_2s_1] &= (y_{62}\{\,+\alpha_{32}y_{21}\,\})[s_1] - \big(\alpha_{31}y_{32}+\alpha_{32}(y_{11}+y_{21})\big)[s_2s_1] + \alpha_{11}\alpha_{32}[s_1s_2s_1] \\
[s_1s_2s_1][s_1s_2s_1s_2s_1] &= \{\,-(y_{43}+y_{53})[s_1]\,\} + (y_{33}+y_{43}+y_{53})[s_2s_1] - \alpha_{11}(1+y_{11}+y_{22})[s_1s_2s_1] \\
[s_1s_2s_1][s_2s_1s_2s_1s_2] &= \{\,(y_{11}+y_{21})[1] - (y_{11}+y_{21})[s_1] - (y_{11}+y_{21})[s_2]\,\} \\
&\quad + y_{11}[s_1s_2] + (y_{11}+y_{21})[s_2s_1] - \alpha_{11}[s_1s_2s_1]
\end{aligned}
$$

$$
\begin{aligned}
[s_2s_1s_2]^2 &= -\alpha_{21}(y_{21}+y_{42})[s_2] + \big(\alpha_{11}\alpha_{21}y_{31}+\alpha_{21}\alpha_{31}y_{10}\big)[s_1s_2] - \alpha_{10}\alpha_{21}\alpha_{31}[s_2s_1s_2] \\
[s_2s_1s_2][s_1s_2s_1s_2] &= y_{53}[s_2] - \big(\alpha_{21}y_{31}+\alpha_{11}\alpha_{21}\alpha_{32}y_{21}\big)[s_1s_2] + \alpha_{21}\alpha_{31}[s_2s_1s_2] \\
[s_2s_1s_2][s_2s_1s_2s_1] &= \{\,-\big(y_{51}+y_{52}+\alpha_{31}y_{10}\big)[1]\,\} + (y_{41}\{\,+\alpha_{31}y_{10}\,\})[s_1] + (y_{42}+y_{52}\{\,+\alpha_{31}y_{10}\,\})[s_2] \\
&\quad - (\alpha_{11}y_{31}+\alpha_{31}y_{10})[s_1s_2] - \alpha_{31}y_{10}[s_2s_1] + \alpha_{10}\alpha_{31}[s_2s_1s_2] \\
[s_2s_1s_2][s_1s_2s_1s_2s_1] &= \{\,(y_{31}+y_{32}+y_{42})[1] - (y_{31}+y_{32})[s_1] - (y_{31}+y_{32}+y_{42})[s_2]\,\} \\
&\quad + (y_{31}+y_{32})[s_1s_2] + y_{31}[s_2s_1] - \alpha_{31}[s_2s_1s_2] \\
[s_2s_1s_2][s_2s_1s_2s_1s_2] &= y_{31}[s_1s_2] - \alpha_{31}[s_2s_1s_2]
\end{aligned}
$$

$$
\begin{aligned}
[s_1s_2s_1s_2]^2 &= \{\, -y_{43}[s_2]\,\} + (y_{32} + y_{42}\{\, +\alpha_{01}y_{21} + \alpha_{32}y_{11}\,\})[s_1s_2] \\
&\quad - \big(\alpha_{01}(y_{11}+y_{21}) + \alpha_{31}(y_{01}+y_{11})\big)[s_2s_1s_2] + \alpha_{01}\alpha_{11}[s_1s_2s_1s_2] \\
[s_1s_2s_1s_2][s_2s_1s_2s_1] &= \{\, (y_{21}+y_{31}+y_{32}+y_{42}+\alpha_{11})[1] \\
&\quad - (y_{21}+y_{31}+y_{32}+\alpha_{11})[s_1] - (y_{21}+y_{31}+y_{32}+y_{42}+\alpha_{11})[s_2]\,\} \\
&\quad + (y_{31}+y_{42}\{, +\alpha_{11}\,\})[s_1s_2] + (y_{21}+y_{31}\{\, +\alpha_{11}\,\})[s_2s_1] \\
&\quad - \alpha_{11}[s_1s_2s_1] - \alpha_{31}[s_2s_1s_2] \\
[s_1s_2s_1s_2][s_1s_2s_1s_2s_1] &= \{\, -(y_{01}+y_{11}+y_{21}+y_{22}+y_{32})[1] \\
&\quad + (y_{01}+y_{11}+y_{21}+y_{22})[s_1] + (y_{01}+y_{11}+y_{21}+y_{22}+y_{32})[s_2] \\
&\quad - (y_{01}+y_{11}+y_{21}+y_{22})[s_1s_2] - (y_{01}+y_{11}+y_{21})[s_2s_1]\,\} \\
&\quad + y_{01}[s_1s_2s_1] + (y_{01}+y_{11}+y_{21})[s_2s_1s_2] - \alpha_{01}[s_1s_2s_1s_2] \\
[s_1s_2s_1s_2][s_2s_1s_2s_1s_2] &= \{\, -y_{21}[s_1s_2]\,\} + (y_{11}+y_{21})[s_2s_1s_2] - \alpha_{11}[s_1s_2s_1s_2]
\end{aligned}
$$

$$
\begin{aligned}
[s_2s_1s_2s_1]^2 &= \{\, -y_{52}[s_1] + (y_{42}+y_{52})[s_2s_1]\,\} - (\alpha_{11}y_{31} + \alpha_{31}y_{10})[s_1s_2s_1] + \alpha_{10}\alpha_{31}[s_2s_1s_2s_1] \\
[s_2s_1s_2s_1][s_1s_2s_1s_2s_1] &= \{\, y_{42}[s_1] - (y_{31}+y_{41})[s_2s_1]\,\} + (y_{31}+y_{32})[s_1s_2s_1] - \alpha_{31}[s_2s_1s_2s_1] \\
[s_2s_1s_2s_1][s_2s_1s_2s_1s_2] &= \{\, -y_{10}[1] + y_{10}[s_1] + y_{10}[s_2] - y_{10}[s_1s_2] - y_{10}[s_2s_1]\,\} \\
&\quad + y_{10}[s_1s_2s_1] + y_{10}[s_2s_1s_2] - \alpha_{10}[s_2s_1s_2s_1]
\end{aligned}
$$

$$
\begin{aligned}
[s_1s_2s_1s_2s_1]^2 &= \{\, -y_{32}[s_1] + (y_{22}+y_{32})[s_2s_1] - (y_{11}+y_{21}+y_{22})[s_1s_2s_1]\,\} \\
&\quad + (y_{01}+y_{11}+y_{21})[s_2s_1s_2s_1] - \alpha_{01}[s_1s_2s_1s_2s_1] \\
[s_1s_2s_1s_2s_1][s_2s_1s_2s_1s_2] &= \{\, [1] - [s_1] - [s_2] + [s_1s_2] + [s_2s_1] \\
&\quad - [s_1s_2s_1] - [s_2s_1s_2]\,\} + [s_1s_2s_1s_2] + [s_2s_1s_2s_1]
\end{aligned}
$$

$$
[s_2s_1s_2s_1s_2]^2 = y_{10}[s_1s_2s_1s_2] - \alpha_{10}[s_2s_1s_2s_1s_2]
$$

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